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An investigation of the survival probability for chaotic diffusion in a family of discrete Hamiltonian mappings

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ABSTRACT

Diffusive processes usually model the transport of particles in nonlinear systems. Complete chaos leads to normal diffusion, while mixed phase space gives rise to a phenomenon called stickiness, leading to anomalous diffusion. We investigate the survival probability that a particle moving along a chaotic region in a mixed-phase space has to survive a specific domain. We show along the chaotic part far from islands that an exponential decay describes the survival probability. Nonetheless, when the islands are incorporated into the domain, the survival probability exhibits an exponential decay for a short time, changing to a slower decay for a considerable enough time. This changeover is a signature of stickiness. We solve the diffusion equation by obtaining the probability density to observe a given particle along a specific region within a certain time interval. Integrating the probability density for a defined phase space area provides analytical survival probability. Numerical simulations fit well the analytical findings for the survival probability when the region is fully chaotic. However, the agreement could be better when mixed structure with islands and periodic areas are included in the domain.

1. Introduction

Hamiltonian systems are a class of dynamical systems in classical mechanics that describe the motion of particles or objects based on energy conservation which has been studied for centuries, with seminal contributions from mathematicians such as Lagrange, Jacobi, and Poincaré. In integrable Hamiltonian systems, the motion of particles is predictable and follows regular patterns, allowing for the system's complete solution through algebraic or geometric methods. In this case, the number of constants of motion equals the degrees of freedom marked by the number of canonical variables pairs (e.g., position and momentum).

However, not all Hamiltonian systems are integrable, and the behavior of non-integrable systems can be much more complex and difficult to predict. Generally, Hamiltonian systems are, usually, non-integrable and non-ergodic [1–3]. Non-integrability arises due to non-linearities or perturbations in the Hamiltonian equations, which lead to chaotic behavior and a breakdown of predictability. In this context, non-integrable Hamiltonian systems have become an essential topic of research in modern mathematics and physics, with significant implications in fields such as celestial mechanics [4], plasma physics [5], and quantum mechanics [6,7].

A powerful observable in the study of the statistical properties of the transport is the survival probability, that is, the probability that a diffusing particle remains in a determined region at some time. To study chaotic Hamiltonian systems, we can divide the analysis into two groups [8]. In the case of fully chaotic systems, the decay over time follows an exponential pattern: $P(\tau) \sim e^{-\xi\tau}$ [9]. However, most physically realistic systems are a mixture of chaotic and regular dynamics in the phase space. In these systems, there is the stickiness effect, where the particles tendency to stick to regular regions, resulting in the probability $P(\tau)$ decays by a power law: $P(\tau) \sim \tau^{-\gamma}$ [10]. This statistic is associated with anomalous transport [11,12], that is, a diffusive process where the mean square displacement increases with a power β of τ that is not equal to 1 $\left\langle \left(\Delta_\tau x\right)^2 \right\rangle \sim \tau^{\beta}$, where $x(\tau)$ represents dynamical variable which presents anomalous diffusion [13,14]. This observation has motivated the adoption of appropriate statistical theories like fractional kinetics [15–17] and continuous-time random walk (CTRW) [18–20] to describe such systems. Furthermore, both exponents can be related by relationship $\gamma + \beta \approx 3$ [9]. This paper considers a family of two-dimensional, nonlinear, and area-preserving maps described by the angle θ and action I variables. The nonlinear term is given by a sine function controlled by a parameter ϵ. For $\epsilon = 0$,

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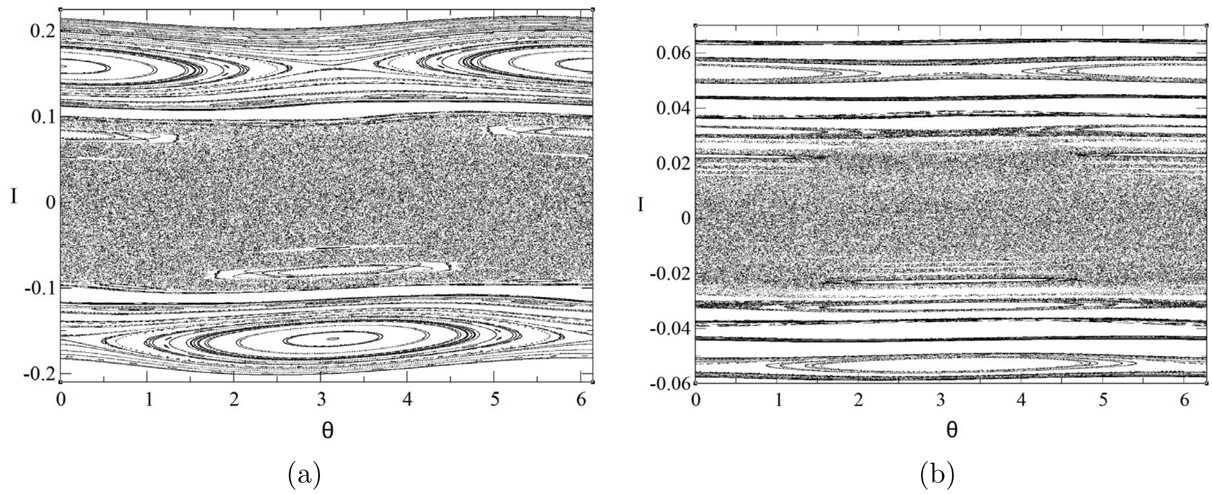

Fig. 1. Plot of the phase space obtained from iteration of (4) considering $\gamma = 1$ and: (a) $\varepsilon = 10^{-2}$ and; (b) $\varepsilon = 10^{-3}$.

the system is integrable, while it is non-integrable for $\varepsilon \neq 0$. As we will see, for the case of $\varepsilon \neq 0$, the phase space is mixed, containing chaotic dynamics, periodic islands, and a set of invariant tori working as barriers do not let chaotic particles moving in the chaotic sea to cross through them.

Our interest in this paper is understanding and describing how the periodic islands influence the survival probability that a particle moving in a chaotic domain has to live inside such a domain. To do that, we solve analytically the diffusion equation for a set of particles moving along the chaotic dynamics and determine, so far, for the first time in the present model, an analytical expression that describes the survival probability for full chaotic dynamics. We show that if the domain is fully chaotic without periodic structures, the survival probability is characterized by an exponential decay. However, when the environment includes regular regions such as stability islands, the particles experience local trapping denoted as stickiness [21–23], which play an important role in Hamiltonian systems affecting the statistical properties of the transport, consequently, diffusion [24–26]. While for the fully chaotic domain, the diffusion is normal, with the occurrence of stickiness, it turns out to be anomalous. Finally, we demonstrate that the survival probability exhibits a scaling invariance concerning the escape for values of weight L less than the action value of the first island of phase space from the sea of chaos.

The paper is organized as follows. In Section 2, we describe the mapping and construct the system's phase space. Section 3 is dedicated to the study of the survival probability of a particle starting in a specific region to survive that domain. We show that the survival probability obtained analytically from the diffusion equation for the fully chaotic domain agrees with the numerical simulations. However, as soon as the periodic domains are included in the region, the agreement between the two procedures is not observed anymore, giving clear evidence that stickiness is affecting diffusion. Furthermore, we show that the survival probability present scaling invariance for completely chaotic survival regions. Section 4 draws final remarks, conclusions, and perspectives.

2. The mapping and the phase space

A Hamiltonian system is considered integrable if it can be reduced to (or integrated by) quadratures [27]. More precisely, if one can construct a simpletic coordinate transformation $(I, \theta) \mapsto (q(I, \theta), p(I, \theta))$, where $I = \{I_1, \ldots, I_n\} \in B \subset \mathbb{R}^n$ with B being an open set, and $\theta \in \{(\phi_1, \ldots, \phi_n) \bmod 2\pi\}$, that is, θ is inside of a n-dimensional torus. These coordinates are called of Action–Angle coordinates and the Hamiltonian becomes $H(q(I, \theta), p(I, \theta)) \equiv K(I)$ with Hamilton's equation given by

$$\dot{I} = -\frac{\partial K(I)}{\partial \theta} = 0$$
$$\dot{\theta} = \frac{\partial K(I)}{\partial I} := \omega(I)$$

However, Hamiltonian systems are typically non-integrable, and non-ergodic [1–3]. Taking this as a motivation, we considered a perturbed Hamiltonian system written as

$$H\left(I_1, \theta_1, I_2, \theta_2\right) = H_0\left(I_1, I_2\right) + \varepsilon H_1\left(I_1, \theta_1, I_2, \theta_2\right), \quad (1)$$

where H_0 corresponds to the integrable part while H_1 denotes the non-integrable part which is controlled by the parameter ε. For $\varepsilon = 0$, the system is integrable since both energy and action are constants, which are not observed for $\varepsilon \neq 0$. Since H does not depend explicitly on time [28], the energy $H = E$ is a constant. Hence, eliminating I_2 from H allows us to write $H = H\left(I_1, \theta_1, \theta_2, E\right)$. This leads the four-dimensional flux of solution in the phase space to be reduced to a three-dimensional flux due to the energy preservation. To characterize the dynamics in terms of a mapping, we consider a Poincaré section (an interception of the flux by a constant plane) defined by the plane $I_1 \times \theta_1$ considering θ_2 as constant $(\bmod\ 2\pi)$. It then leads the dynamics from 3-D flow to an application in a 2-D mapping that can be described by a generic mapping of the form (1)):

$$\begin{cases} I_{n+1} = I_n + \epsilon h\left(\theta_n, I_{n+1}\right) \\ \theta_{n+1} = \left[\theta_n + K\left(I_{n+1}\right) + \epsilon p\left(\theta_n, I_{n+1}\right)\right] \quad \bmod (2\pi) \end{cases} \quad (2)$$

where the index n indicates the number of iteration of mapping and $h\left(\theta_n, I_{n+1}\right)$, $K\left(I_{n+1}\right)$, $p\left(\theta_n, I_{n+1}\right)$ are nonlinear functions of their variables. Note that the mapping (2) preserves the area if the following condition is satisfied:

$$\det J = 1 \Longleftrightarrow \frac{\partial p\left(\theta_n, I_{n+1}\right)}{\partial \theta_n} + \frac{\partial h\left(\theta_n, I_{n+1}\right)}{\partial I_{n+1}} = 0 \quad (3)$$

where J is the Jacobian Matrix of (2).

Considering $p\left(\theta_n, I_{n+1}\right) = 0$ and $h\left(\theta_n, I_{n+1}\right) = \sin\left(\theta_n\right)$, we have that several dynamical systems well known in the literature are obtained when specific choices are made:

- $K\left(I_{n+1}\right) = I_{n+1}$, recovering the standard mapping [29,30]
- $K\left(I_{n+1}\right) = 2/I_{n+1}$, leading to the static wall approximation of the Fermi–Ulam model [31];
- $K\left(I_{n+1}\right) = \alpha I_{n+1}$, giving the simplified bouncer model [32], with α as a constant;

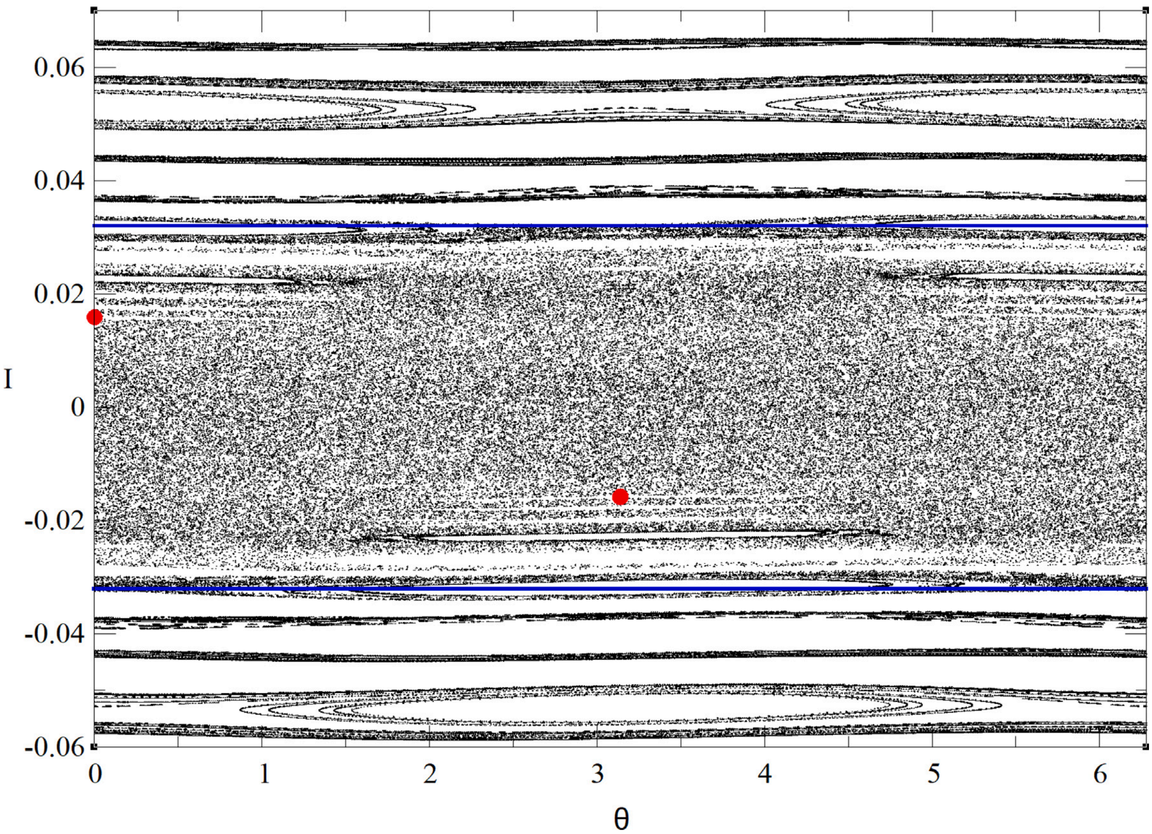

Fig. 2. Phase space of the mapping (4) with $\gamma = 1$ and $\epsilon = 10^{-3}$, where the first elliptic fixed points marked with red bullets, these coordinates ($(0, I_{fii})$ and $(\pi, -I_{fii})$) are the centers of first period islands from the origin of phase space, and is identified by the blue curves, the first invariant spanning curves.

In our investigation, we consider the following generalized mapping:

$$\begin{cases} I_{n+1} = I_n + \epsilon \sin(\theta_n) \\ \theta_{n+1} = \left[\theta_n + \frac{1}{|I_n+1|^\gamma}\right] \quad \bmod (2\pi) \end{cases} \tag{4}$$

where $\gamma \geq 0$ is a control parameter controlling the speed of the divergence of θ in the limit of I sufficiently small. Since the variable θ is bounded $\theta \in [0, 2\pi]$ the limit when I goes to 0 turns θ_{n+1} and θ_n uncorrelated. On the other hand, when I grows, regularity appears in the phase space, leading to periodic islands and invariant curves. Fig. 1 shows the phase space of (4) for $\gamma = 1$ and: (a) $\epsilon = 10^{-2}$; (b) $\epsilon = 10^{-3}$.

Spanning invariant curves are those ones cross the space phase, consequently, the first invariant spanning curves limit the size of chaotic sea and prevent the diffusion to be unlimited. The position of first invariant spanning curve [29,33] can be approximated by

$$I_{fisc} = \left(\frac{\gamma\epsilon}{0.9716}\right)^{\frac{1}{\gamma+1}}$$

The elliptic fixed points are given by

$$(\theta^*_{ellip}, I^*_{ellip}) = \begin{cases} \left(0, \left(\frac{1}{2m\pi}\right)^{\frac{1}{\gamma}}\right) \\ \left(\pi, -\left(\frac{1}{2m\pi}\right)^{\frac{1}{\gamma}}\right) \end{cases}, \qquad \text{for } m < \frac{1}{2\pi}\left(\frac{4}{\epsilon\gamma}\right)^{\frac{\gamma}{\gamma+1}}$$

These points represent the center of instability islands, in this way we can characterize the position of the first island of phase space from the sea of chaos, which we are denoted by I_{fii}.

From now on we consider $\gamma = 1$ and $\epsilon = 10^{-3}$, and for this situation we have the following values $I_{fisc} = 0.03208$ and $I_{fii} = 0.01592$. In Fig. 2 we displayed these two observable.

3. Survival probability

Three important points are discussed in this section. The first one is the analytical solution of the diffusion equation by imposing absorbing boundary conditions, giving the probability density to obtain a particle at a certain domain of the phase space at a time n. The second is the integration of the probability density limited to a finite domain in the phase leading to the survival probability. The third is the scaling invariant of the survival probability for chaotic domain. To start, we consider the diffusion equation

$$\frac{\partial P(I,n)}{\partial n} = D\frac{\partial^2 P(I,n)}{\partial I^2}, \tag{5}$$

The function $P(I, n)$ is the probability density to observe a particle with an action I at a time n with D representing the diffusion coefficient. The veracity of Eq. (5) is plausible due the averaged squared action $\langle I^2\rangle$, in chaotic domain, is linearity dependent of the number of interactions n, i.e. $\langle I^2\rangle \propto n$ characterizing the process of normal diffusion. Assuming statistical independence between I and θ at the chaotic diffusion, we obtain $D = \frac{(\Delta I)^2}{2} = \frac{\epsilon^2}{4}$.

In practical situations involving diffusion, we often focus on scenarios where the diffusing particle is confined within a finite area. The specific boundary conditions that apply, which are determined by the physical context, can result in various outcomes. Interestingly, the solution to the diffusion equation is highly influenced by the imposed boundary conditions. A significant scenario is the diffusive process operating under absorbing boundary conditions. In this situation, when the diffusing particle encounters the perfect *"absorbers"* located at the endpoints $I = \pm L$, it gets absorbed or annihilated at the first contact, thus marking the end of the diffusion process, then these endpoints

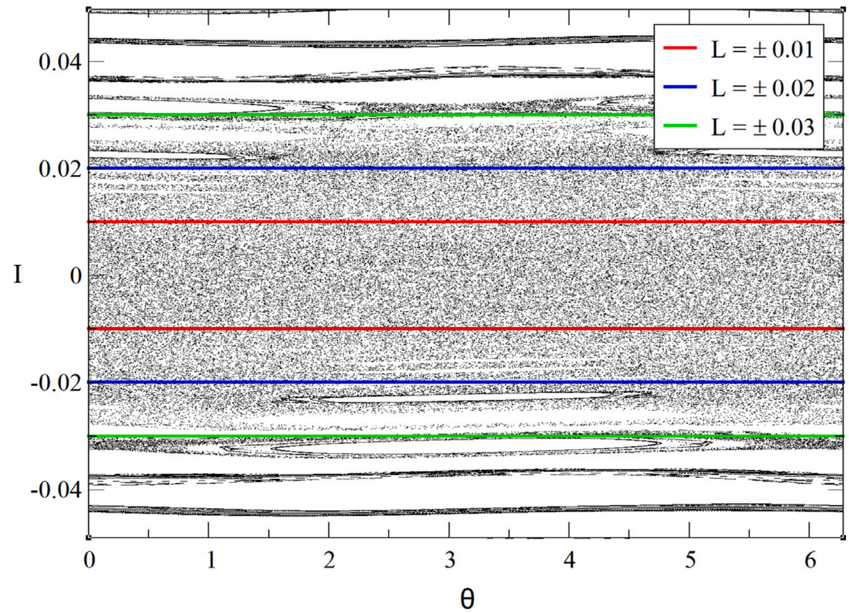

Fig. 3. Survival region L, which we will consider in the calculation of the probability a particle survives along the chaotic dynamics inside this region.

delimited the region of investigate. In this way, the suitable boundary conditions that account for the presence of perfectly absorbing barriers situated at $I = \pm L$ are

$$\begin{cases} P(I,0) = \delta(I - I_0), & \text{(a)} \\ P(-L,n) = P(L,n) = 0. & \text{(b)} \end{cases} \tag{6}$$

We use the technique of separation of variables to solve the diffusion equation. We, therefore, assume that $P(I,n) = X(I)Y(n)$, where X and Y are functions to be determined. Considering a symmetric property $P(I,n) = P(-I,n)$ for the problem, we obtain a countable numbers of harmonic functions for $X(I)$ and $Y(n)$, which are given by

$$\begin{cases} X(I) = c_m \cos\left(\dfrac{(2m+1)\pi I}{2L}\right), \\ Y(n) = \gamma_m \exp\left(-\dfrac{(2m+1)^2\pi^2}{4L^2} Dn\right), \end{cases} \tag{7}$$

where c_m, γ_m are constants and $m \neq 0$ is an integer. Then, by the superposition principle, and calling $\alpha_m = c_m\gamma_m$, we have that

$$P(I,n) = \sum_{m=0}^{\infty} \alpha_m \cos\left(\frac{(2m+1)\pi I}{2L}\right) \exp\left(-\frac{(2m+1)^2\pi^2}{4L^2} Dn\right). \tag{8}$$

Now, imposing the initial condition (6), we obtain

$$\delta(I - I_0) = \sum_{m=0}^{\infty} \alpha_m \cos\left(\frac{(2m+1)\pi I}{2L}\right), \tag{9}$$

which is the Fourier Series for delta function with period $2L$, where its coefficients are given by

$$\alpha_m = \frac{1}{L}\int_{-L}^{L} \delta(I - I_0) \cos\left(\frac{(2m+1)\pi I}{2L}\right) dI = \frac{1}{L}\cos\left(\frac{(2m+1)\pi I_0}{2L}\right). \tag{10}$$

Then, the solution of the diffusion equation is

$$P(I,n) = \frac{1}{L}\sum_{m=0}^{\infty} \cos\left(\frac{(2m+1)\pi I_0}{2L}\right) \cos\left(\frac{(2m+1)\pi I}{2L}\right) \times \exp\left(-\frac{(2m+1)^2\pi^2}{4L^2} Dn\right). \tag{11}$$

Note that the series (11) converges because the absolute value of term m is less than $\exp(-cm^2)$ for some $c > 0$, which converges. It is also interesting to observe that the probability goes to zero as time diverges, allowing all particles to be absorbed by the boundaries.

Let now move on and investigate the behavior of the survival probability [34]. It is written as

$$S_{(-L,L);I_0}(n) = \int_{-L}^{L} P(I,n)dI, \tag{12}$$

where $0 < L < I_{fisc}$ defines a domain in the phase space where the particle can move and diffuse. The initial condition at $n = 0$ is $I = I_0 \ll \varepsilon$, as shown in Fig. 3.

From the expression for probability $P(I,n)$, we end up with

$$\begin{aligned} S_{(-L,L);I_0}(n) &= \frac{1}{L}\sum_{m=0}^{\infty} \cos\left(\frac{(2m+1)\pi I_0}{2L}\right)\left(\int_{-L}^{L}\cos\left(\frac{(2m+1)\pi I}{2L}\right)dI\right) \\ &\quad \times \exp\left(-\frac{(2m+1)^2\pi^2}{4L^2} Dn\right) \\ &= \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)}\cos\left(\frac{(2m+1)\pi I_0}{2L}\right) \\ &\quad \times \exp\left(-\frac{(2m+1)^2\pi^2}{4L^2} Dn\right). \end{aligned} \tag{13}$$

Since we now have the analytical expression of the survival probability, let us investigate the influence of the stability islands on the behavior of the diffusing particles. To do that, we must numerically follow a set of initial conditions in the chaotic domain. We then calculate a histogram showing the escape time for the particles, $H_N(n)$. It informs the number of initial conditions from an ensemble of N particles that escaped through the borders at the iterated n.

$H_N(n)$ is calculated as follows. Consider a set of N distinct initial conditions in the desired region. Each one of them is evolved. When the particle reaches the borders either at $-L$ or L, the number of iterates up to that instant is recorded. A new initial condition is started, and the procedure is repeated until the entire ensemble is exhausted. The sum of all initial conditions that escaped the domain in n iterations is stored in $H_N(n)$. We considered an ensemble of $N = 10^6$ initial conditions given by the set $A = \{(I_0, \frac{2\pi j}{10^6})\}$ for $I_0 = 10^{-5}\varepsilon$ and $j = 1,\ldots,10^6$. Each initial condition was allowed to evolve at a maximum time of 10^5 if the particle did not escape before. The histogram of escaping particles $H_N(n)$ is shown in Fig. 4.

Now, we will investigate how the transport occurs for some values of L, where it is displayed in Fig. 5. The color range denotes the number

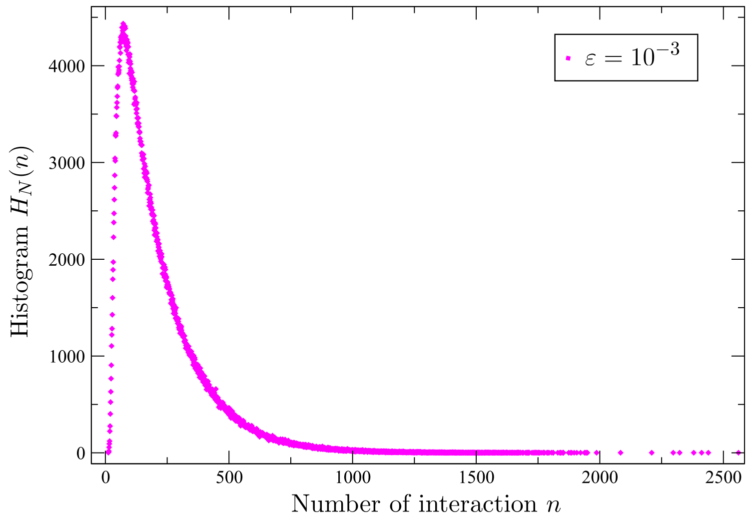

Fig. 4. The histogram of escaping particles $H_N(n)$ for parameter $\varepsilon = 10^{-3}$.

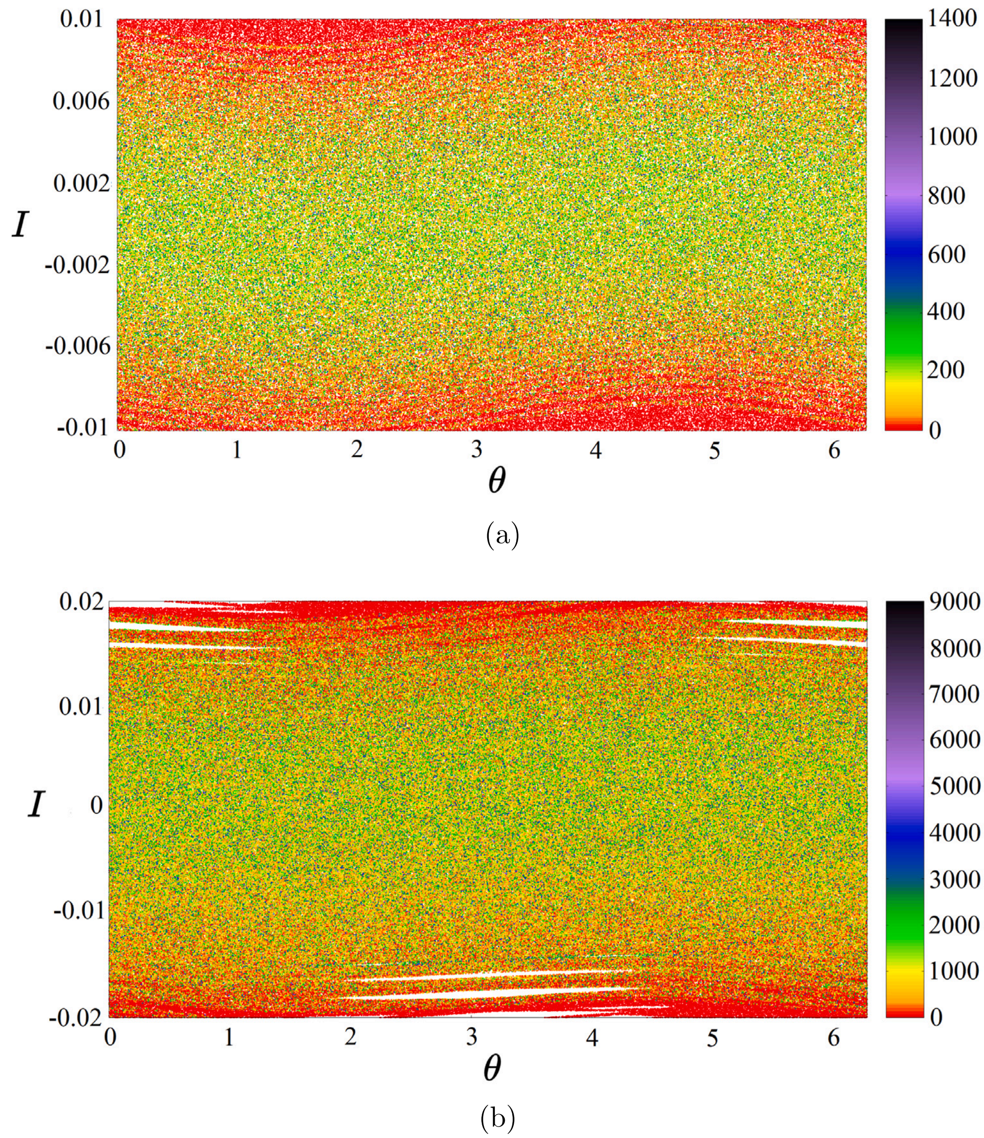

Fig. 5. Plot of the time evolution of initial conditions to can escape from a survival region (a) $L = 0.01$ and (b) $L = 0.02$. Essentially, black, blue and purple indicates long interaction until the particle escapes, while red indicates that it present a fast escape. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

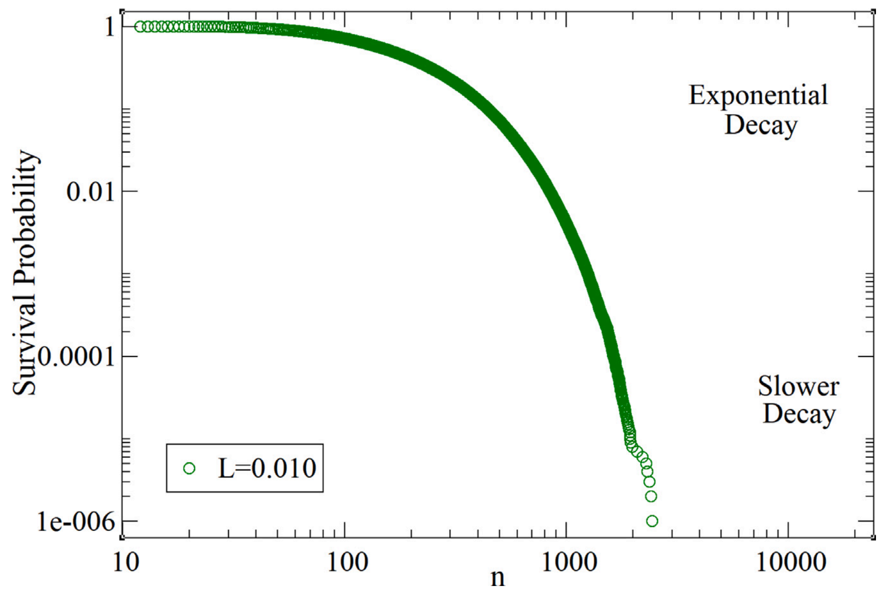

Fig. 6. Survival Probability of a region with weight $L = 0.010$. Initially, the curve has an exponential decay, according Eq. (15), while in your "final tail", it experiences a slower decay which is due stickiness effect.

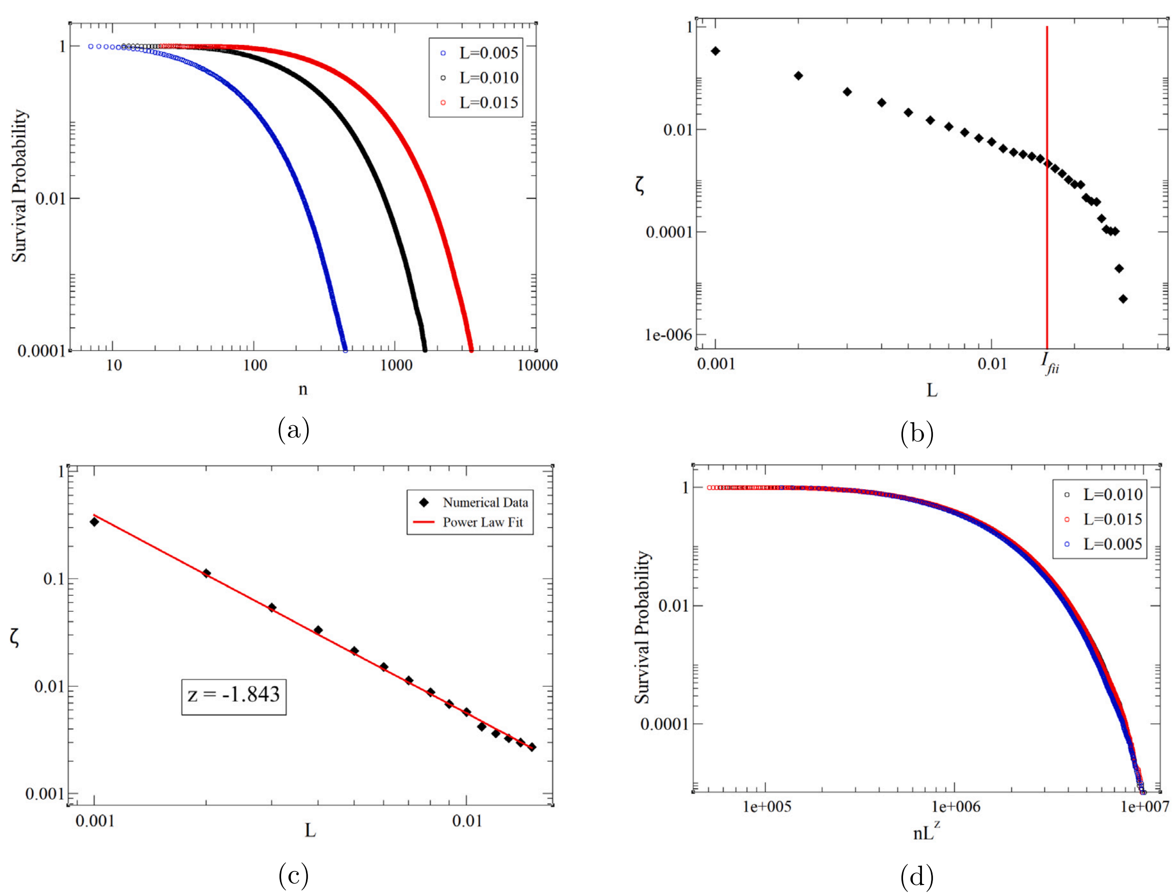

Fig. 7. (a) Plot of the survival probability curves $S_{(-L,L);I_0}(n)$ for $L = 0.005$, $L = 0.010$ and $L = 0.015$. We have the a study of ζ as a function of L in (b) and for values less than I_{fii} of the variable in (c). In your analysis we obtain that a power law $\zeta \propto L^z$ furnishes a good fitting, with an exponent $z = -1.843$ for weight described in (c). At last, in (d) we show the scaling invariance of the survival probability curves for a small height L.

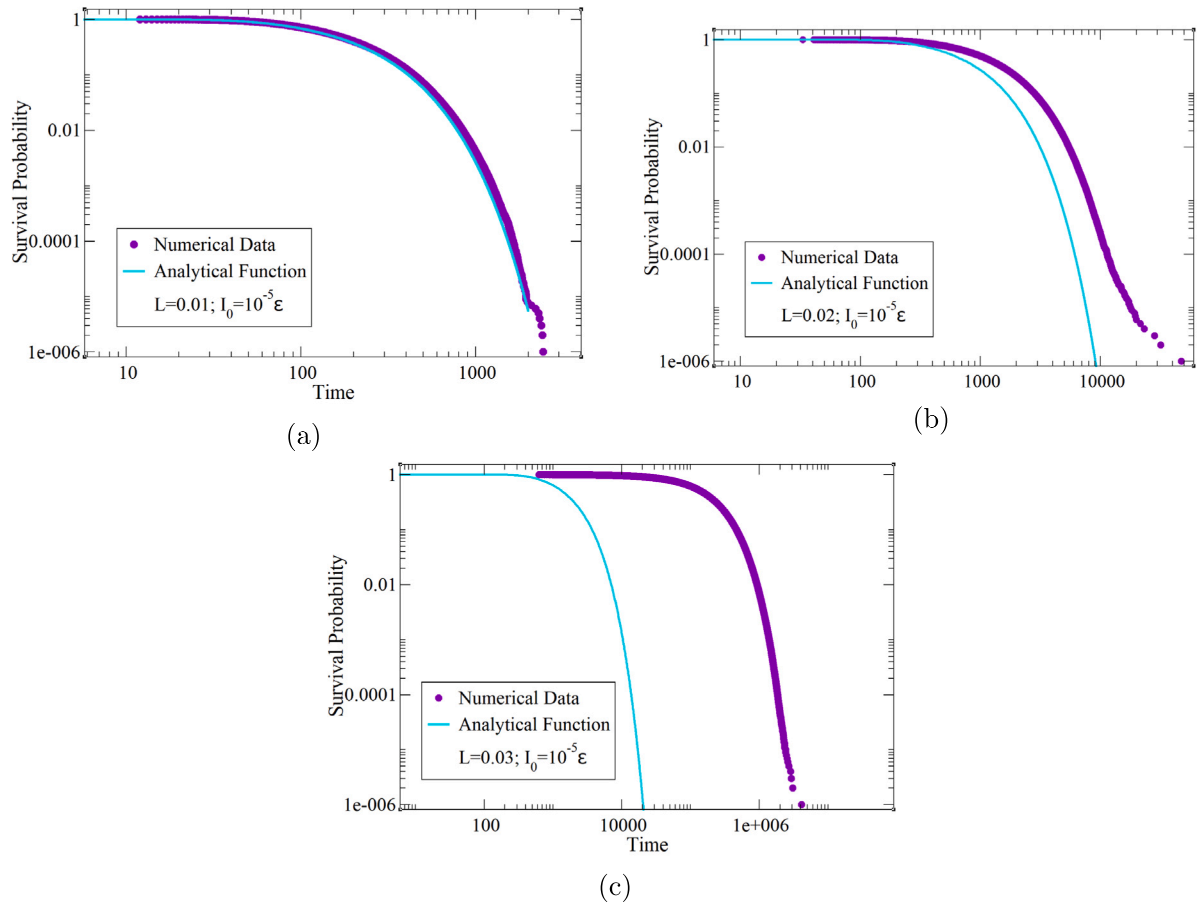

Fig. 8. The figure describe the numerical and analytical (obtained from diffusion equation) survival Probability of a particle for a region L, i.e., the probability a particle survive along the chaotic dynamics inside a given domain L without escaping such region.

of interaction that the initial condition had until to escape from the survival probability. Basically, it can be interpreted as red indicating fast escape, blue/black denoting long time dynamics. One can see that even near of $I = 0$, where we have chaos, there are some initial condition (e.g. blues points) which escapes for a long time, evidencing that it was trapped. However, as the most of initial condition in this region escape fast (Nearby of $I = 0$, Fig. 5 has more green points), thus we considered the average, the diffusion is normal and agree with the analytical consideration as illustrated in Fig. 8(a).

Finally, the survival probability is given by

$$S_{(-L,L);I_0}(n) = 1 - \frac{1}{M}\sum_{n=1}^{M} H_N(n). \tag{14}$$

Fig. 6 shows the survival probability for $L = 0.01$. It is known in the literature [35–38] that the decay of the survival probability for strongly chaotic systems is usually exponential and is given by

$$S_{(-L,L);I_0}(n) \propto A \exp -\zeta n \tag{15}$$

where A is a positive constant and ζ is the decay rate. However, the decay is slower for systems with mixed phase space and was observed either as a power law [39,40], or as a stretched exponential [41]. The difference is the occurrence of stickiness where a particle passes close to periodic regions and becomes trapped for a certain time that may be, eventually, long. The effect of the stickiness is seen in Fig. 6.

As expected, the behavior of $S_{(-L,L);I_0}(n)$ depends on L, which defines the domain the particle diffuses, as shown in Fig. 7(a). Since the exponential decay ζ is related to L, Fig. 7(b) shows a plot of ζ $vs.$ L. We notice that for values of height L which are less than I_{fii}, the exponential decay is described by a power law L. A fitting considering $L \in [0.01, 0.15]$ leads to $\zeta \propto L^z$, with $z = -1.843$ as shown in Fig. 7(c). The knowledge of z allows us to re-scale the horizontal axis as $n \to nL^z$ producing an overlap of the curves plotted in Fig. 7(a) onto a single and universal plot indicating that the exponential decay rate is scaling invariant in small survival regions, as shown in Fig. 7(d).

We now compare the analytical and numerical results for different values of L. Fig. 8 shows a plot of the survival probability as a function of time for three different values of L. The agreement between these two procedures is good only in regions where stickiness is not observed. The influence of the islands affects the survival probability curve, allowing the particles to live longer in the chaotic domain. Notice that, for values of L less than I_{fii} we have the analytical accord with numerical results for survival probability.

4. Conclusion

An investigation of the survival probability shows us that it exhibits a exponential decay rates, followed by power law tails, evidencing the influence of sticky orbits in the dynamics. Furthermore, we obtained a analytical expression for it, which has a good agreement for small values of L. Which is plausible, since in this deduction we are considering that the diffusion is normal, which describes chaotic orbits, then, as our initial conditions presents a small value for the action, i.e., it is essentially in chaos regime, we have that it diffuse in a normal way. However when we consider bigger survival region, this region has stability islands, consequently more stickiness orbits, thus the trapping effect in the diffusion is more evident, modifying the type of process.

Last but not least, we show that the survival probability are scaling invariant with respect to the weight of escape L for values less that I_{fii}. The reason is that for these values we do not have periodic island inside a region L and as trapping domain are associated to these islands, we do not have stickiness effect for values of $L < I_{fii}$, then we not perturb the diffusion then the survival probability can be described by a analytical form and present scaling invariant. Beyond that, under this configuration we study the escape (decay) rate, denoted as ζ, it can be described by a algebraic law of L, with a exponent $z = -1.843$. However, as we show that the system present diffusion in this form, the escape rate can be calculated by comparing it with the diffusion equation. In this case, the calculation suggests that ζ is equal to $\frac{D\pi^2}{4}L^{-2}$, specifically corresponding to the $m = 0$ term of Eq. (13) which exhibits the slowest decrease. Therefore, the exponent z, which is measured to be -1.843 instead of -2, indicates that the diffusion equation approach provides a reasonably accurate estimation.

As a perspective for future works, we intend to study the transport between ellipse region around stability island and investigate how the survival probability behaves in this situation, also we intent to changing the size and location of the escape regions to seek and understand their role in the decay rate of the survive probability, as done in [42]. Furthermore, as we know that for bigger values of L, the stickiness effect is more evident, consequently, the diffusion is anomalous, it is necessary to utilize the theory of fractional partial differential equations to obtain a analytical description of the model.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Daniel Borin reports financial support was provided by State of Sao Paulo Research Foundation. Edson Denis Leonel reports financial support was provided by State of Sao Paulo Research Foundation. Edson Denis Leonel reports financial support was provided by National Council for Scientific and Technological Development.

Data availability

No data was used for the research described in the article

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